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On the Hecke algebra of a type for unramified p-adic unitary groups

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ABSTRACT

Let F be a non-archimedean local field of odd residue characteristic equipped with a galois involution with fixed field F_0, and let G be a symplectic group over F or an unramified unitary group over F_0. For a simple type (J_P, λ_P) in G, which is a G-cover, associated to a certain simple stratum, we show that the corresponding Hecke algebra $\mathcal{H}(G, \lambda_P)$ is generic.

Introduction

Let F be a non-archimedean local field, and G the F-rational points in a connected reductive group defined over F. By a theorem of Bersntein in [1], the category $\mathcal{R}(G)$ of complex smooth representations of G is decomposed into an infinite product of full subcategories, called Bernstein blocks, which are determined by the inertial classes of pairs (M, π) consisting of a Levi subgroup M of G and an irreducible supercuspidal representation π of M. By the theory of types, due to Bushnell and Kutzko [3], up to conjugation by G, each Bernstein block corresponds to a unique pair (K, ρ), called a type, which consists of a compact open subgroup K of G and an irreducible representation ρ of K, and it is equivalent to a category of unital left modules over the ρ-spherical Hecke algebra $\mathcal{H}(G, \rho)$.

Let N be an integer $\geqslant 2$, and G the general linear group $GL(N, F)$ over F. In Bushnell and Kutzko [2], such a pair (J, λ), called a simple type, is constructed for $G = GL(N, F)$, and it is proved that it determines a pair (M, π) for G as above. For a parabolic subgroup P of G with Levi factor M,

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pairs (J_P, λ_P) for G and (J_M, λ_M) for M are naturally obtained from (J, λ). The pair (J_M, λ_M) is a supercuspidal type in M for the inertial class of (M, π), and (J_P, λ_P) is a G-cover of (J_M, λ_M). It follows from [3] that (J_P, λ_P) is a type in G for the inertial class of (M, π). The Hecke algebra $\mathcal{H}(G, \lambda_P)$ is isomorphic to a certain affine Hecke algebra of type A.

Let G be a symplectic group $Sp_{2N}(F)$ over F with residue characteristic not 2, and P the Siegel parabolic subgroup of $G = Sp_{2N}(F)$ with Levi factor $M \simeq GL(N, F)$. From a given irreducible supercuspidal representation π_0 of $GL(N, F)$ that is self-dual, a type in G, indeed a G-cover, for the inertial class determined by (M, π_0) is constructed in Blondel [5]. These results are generalized to a general Levi subgroup of G under certain hypotheses, and the corresponding Hecke algebra is computed as a convolution algebra, in [6].

By using the result of [5], a certain cover (J_P, λ_P) is constructed for a Levi subgroup M of an unramified unitary group G, which is similar to the one of [6], in [13]. Independently of this result, this type is obtained from the general results of Stevens [19], in which it is implicit.

In this paper, for the type (J_P, λ_P) in the unramified unitary group G, it is proved that the Hecke algebra $\mathcal{H}(G, \lambda_P)$ is generic, i.e., there is an affine Weyl group $\boldsymbol{W}_m$ with a set of generators $\{s_0, s_1, \ldots, s_m\}$, for some positive integer m, such that the Hecke algebra $\mathcal{H}(G, \lambda_P)$ has a basis $\{T_w \mid w \in \boldsymbol{W}_m\}$ as a vector space over the field $\mathbb{C}$ of complex numbers, where T_w is a function in $\mathcal{H}(G, \lambda_P)$ supported on a single double (J_P, J_P)-coset corresponding to $w \in \boldsymbol{W}_m$, and such that the following relations are satisfied: for $w \in \boldsymbol{W}_m$ and $i \in \{0, 1, \ldots, m\}$, we have

$$T_w * T_{s_i} = T_{ws_i}, \quad \text{if } \ell(ws_i) > \ell(w),$$
$$T_{s_i}^2 = a_i T_{s_i} + b_i 1, \quad a_i, b_i \in \mathbb{C},\ b_i \neq 0,$$

where ℓ is the length function on $\boldsymbol{W}_m$.

The contents of this paper are as follows: In Section 1, we review briefly the results of [13], which are rewritten by the terminology of [19], and recall Proposition 1 of [6]. In Section 2, we define an auxiliary unitary group, and in Section 3, we observe the centralizer in G of a certain simple element of the Lie(G). In Section 4, using the results of Sections 2 and 3, we calculate convolution products in the Hecke algebra $\mathcal{H}(G, \lambda_P)$ to obtain the above main result (Theorem 4.3).

1. A type for unramified p-adic unitary groups

1.1. Let F be a non-archimedean local field equipped with a galois involution $^-$, with the fixed field F_0. Let $\mathfrak{o}_F$ and $\mathfrak{p}_F$ be its maximal order and the maximal ideal of $\mathfrak{o}_F$, respectively, and $k_F = \mathfrak{o}_F/\mathfrak{p}_F$ the residue class field. Let ϖ_F be a uniformizer of F. We assume that the residual characteristic p is not 2 and that F/F_0 is unramified (possibly $F = F_0$).

Let N be an integer $\geqslant 4$. Let V be an N-dimensional vector space over F, and put $A = \mathrm{End}_F(V)$. Let h be a non-degenerate anti-hermitian form on V over F/F_0. We furthermore assume that the anisotropic part of V is zero. Then N must be even. Let $^-$ be the adjoint (anti-)involution on A defined by the form h. Put $\widetilde{G} = \mathrm{Aut}_F(V) \simeq GL(N, F)$. We define γ to be the involution $x \mapsto \bar{x}^{-1}$ on $\widetilde{G}$, and put $\Gamma = \{1, \gamma\}$.

We set $G = \widetilde{G}^{\Gamma} = \{g \in \widetilde{G} \mid h(gv, gw) = h(v, w) \text{ for all } v, w \in V\}$. Then G is a symplectic group over F if $F = F_0$, and is an unramified unitary group over F_0 if $F \neq F_0$. We write $G = U(V, h)$.

An $\mathfrak{o}_F$*-lattice sequence* in V is a function $\Lambda : \mathbb{Z} \to \{\mathfrak{o}_F\text{-lattices in } V\}$ which satisfies (1) $n \geqslant m$ implies $\Lambda(n) \subset \Lambda(m)$, (2) there is a positive integer $e = e(\Lambda)$ (the $\mathfrak{o}_F$*-period*) such that $\Lambda(n+e) = \mathfrak{p}_F \Lambda(n)$ $(n \in \mathbb{Z})$. We say that a $\mathfrak{o}_F$-lattice sequence Λ is *strict*, if $\Lambda(n) \supsetneq \Lambda(n+1)$ $(n \in \mathbb{Z})$.

From an $\mathfrak{o}_F$-lattice sequence Λ in V, we obtain a filtration on A by

$$\mathfrak{a}_n = \mathfrak{a}_n(\Lambda) = \bigl\{x \in A \bigm| x\Lambda(k) \subset \Lambda(k+n)\ (k \in \mathbb{Z})\bigr\} \quad (n \in \mathbb{Z}).$$

We can define a 'valuation' ν_Λ on A by $\nu_\Lambda(x) = \sup\{n \in \mathbb{Z} \mid x \in \mathfrak{a}_n\}$ $(x \in A)$, where we understand that $\nu_\Lambda(0) = \infty$. We note that if Λ is strict, then $\mathfrak{a}_n = \mathfrak{a}_1^n$ $(n \in \mathbb{Z},\ n > 0)$.

From an $\mathfrak{o}_F$-lattice sequence Λ in V, we obtain a compact open subgroup $\widetilde{P} = \widetilde{P}(\Lambda) = \mathfrak{a}_0(\Lambda)^\times$ of $\widetilde{G}$, equipped with a filtration $\widetilde{P}_n = \widetilde{P}_n(\Lambda) = 1 + \mathfrak{a}_n(\Lambda)$ ($n \in \mathbb{Z}$, $n > 0$).

For an $\mathfrak{o}_F$-lattice L in V, we define the *dual lattice* $L^\#$ by $L^\# = \{v \in V \mid h(v, L) \subset \mathfrak{p}_F\}$. An $\mathfrak{o}_F$-lattice sequence Λ in V is called *self-dual*, if there is an integer d such that $\Lambda(k)^\# = \Lambda(d-k)$ ($k \in \mathbb{Z}$). In this case, we can set $P = P(\Lambda) = \widetilde{P}(\Lambda)^\Gamma$, a compact open subgroup of G, and $P_n = P_n(\Lambda) = \widetilde{P}_n(\Lambda)^\Gamma$ ($n > 0$), a filtration on $P = P(\Lambda)$.

1.2. A 4-tuple $[\Lambda, n, 0, \beta]$ is called a *skew simple stratum* in A, if Λ is a self-dual $\mathfrak{o}_F$-lattice sequence in A, n is a positive integer, β is an element in A which satisfies $\overline{\beta} + \beta = 0$ and

(1) the algebra $E = F[\beta]$ is a field;
(2) Λ is an $\mathfrak{o}_E$-lattice sequence (denote it by $\Lambda_{\mathfrak{o}_E}$);
(3) $\nu_\Lambda(\beta) = -n$;
(4) $k_0(\beta, \Lambda) < 0$,

where $\mathfrak{o}_E$ is the maximal order of E and $k_0(\beta, \Lambda)$ is the integer defined by [4, (5.1)].

Let $[\Lambda, n, 0, \beta]$ be a skew simple stratum in A, B the A-centralizer of β, and $E_0 = \{x \in E = F[\beta] \mid \bar{x} = x\}$, where $x \mapsto \bar{x}$ is the involution on E induced by the one on A defined in 1.1. Then by [8, 5.2], there is a non-zero F_0-linear form $\ell : E \to F$ with $\ell(\bar{x}) = \ell(x)$ ($x \in E$), and a non-degenerate E/E_0-skew-hermitian form $\widetilde{h}_\beta$ on V such that the two notions of lattice duality for $\mathfrak{o}_E$-lattices in V given by h and $\widetilde{h}_\beta$ coincide (cf. [19, 2.1]). Set $G_E = B^\times \cap G$. Then G_E is isomorphic to the unitary group of $(V, \widetilde{h}_\beta)$, and we identify $G_E = U(V, \widetilde{h}_\beta)$.

Definition 1.1. (Cf. Definition 2.4 of [13].) A skew simple stratum $[\Lambda, n, 0, \beta]$ in A is called *good*, if there is a skew simple stratum $[\Lambda', n', 0, \beta]$ in A satisfying the following conditions (1)–(4):

(1) Λ' is strict and $\mathfrak{a}_0(\Lambda')$ is principal;
(2) E/E_0 is unramified;
(3) $R = \dim_E(V)$ is even;
(4) there is an integer k such that $L = \Lambda'(k)$ satisfies $L^\# = \varpi_E L$, where ϖ_E is a uniformizer of E,

and $\Lambda = 2\Lambda' - 1$ if $e = e(\Lambda'_{\mathfrak{o}_E})$ is even, $\Lambda = 2\Lambda'$ if e is odd, and $n = 2n'$.

In Definition 1.1, the simple stratum $[\Lambda', n', 0, \beta]$ in A coincides with $[\mathfrak{A}', n', 0, \beta]$, with $\mathfrak{A}' = \mathfrak{a}_0(\Lambda')$, in A of [2, (1.5.5)], and the $\mathfrak{o}_F$-lattice sequence Λ satisfies $\Lambda(k)^\# = \Lambda(1-k)$ ($k \in \mathbb{Z}$) and $e(\Lambda_{\mathfrak{o}_E}) = 2e$, which satisfies the conditions of [19, p. 297].

Proposition 1.2. *(See Proposition 2.7 of [13].) If the conditions* (2), (3) *and* (4) *in Definition* 1.1 *are satisfied, the anisotropic part of* $(V, \widetilde{h}_\beta)$ *is zero.*

Assume that $[\Lambda, n, 0, \beta]$ is a good skew simple stratum in A, and set $e = e(\Lambda_{\mathfrak{o}_E})/2$. We denote by B the A-centralizer of β. We set $m = \lfloor e/2 \rfloor$ the least integer among integers $\geqslant e/2$, and $\mathfrak{b}_k = \mathfrak{a}_k \cap B$ ($k \in \mathbb{Z}$). Then we have an E-decomposition

$$V = \bigoplus_{j=-m}^{m} W^{(j)} \tag{1.1}$$

of V subordinate to the $\mathfrak{o}_E$-order $\mathfrak{B} = \mathfrak{b}_0$ in the sense of [2, (7.1.11)], where we set $j \neq 0$ if e is even.

We write $\widetilde{P}(\Lambda_{\mathfrak{o}_E}) = \widetilde{P}(\Lambda) \cap B^\times$ and $P(\Lambda_{\mathfrak{o}_E}) = \widetilde{P}(\Lambda_{\mathfrak{o}_E}) \cap G$.

Lemma 1.3. *The decomposition* (1.1) *has moreover the following properties*:

(1) *each* $W^{(j)}$ *is an* E*-subspace of* V *of dimension* R/e;
(2) $\Lambda(k) = \bigoplus_{j=-m}^{m} \Lambda^{(j)}(k)$ $(k \in \mathbb{Z})$, *where* $\Lambda^{(j)}(k) = \Lambda(k) \cap W^{(j)}$;
(3) *each* $\widetilde{P}(\Lambda_{\mathfrak{o}_E}^{(j)})$ *is a maximal self-dual* $\mathfrak{o}_E$*-order in* $\mathrm{Aut}_E(W^{(j)})$;
(4) *the orthogonal complement of* $W^{(j)}$ *in* V *is equal to* $\bigoplus_{k \neq -j} W^{(k)}$;
(5) *there exist self-dual* $\mathfrak{o}_E$*-lattice sequences* Λ^{M} *and* Λ^{m} *in* V *such that*

$$\mathfrak{a}_0\big(\Lambda^{\mathsf{m}}\big) \subset \mathfrak{a}_0(\Lambda) \subset \mathfrak{a}_0\big(\Lambda^{\mathsf{M}}\big)$$

with $\mathfrak{b}_0(\Lambda^{\mathsf{M}})$ *maximal and* $\mathfrak{b}_0(\Lambda^{\mathsf{m}})$ *minimal in* B.

Proof. These parts except (5) follow immediately from [2, (5.5.2)] and [13, Proposition 2.7] (see also [19, 5.1]), and part (5) easily from part (4) in Definition 1.1. In fact, Λ^{M} and Λ^{m} can be easily produced from strict self-dual $\mathfrak{o}_E$-lattice sequences in V as in Definition 1.1 (cf. [2, (5.1.12)]). For Λ^{M}, we can choose $\mathfrak{M}_1$ which is defined in [19, 7.2,2]. □

1.3. We assume that $[\Lambda, n, 0, \beta]$ is a good skew simple stratum in A, let $E = F[\beta]$ and B the A-centralizer of β. In [18, 3.2], we define a pair of $\mathfrak{o}_F$-orders $\mathfrak{H}(\beta, \Lambda) \subset \mathfrak{J}(\beta, \Lambda)$ of A, and obtain compact open subgroups $\widetilde{H}(\beta, \Lambda) = \mathfrak{H}(\beta, \Lambda)^{\times}$ and $\widetilde{J}(\beta, \Lambda) = \mathfrak{J}(\beta, \Lambda)^{\times}$ of $\widetilde{G}$ with normal subgroups $\widetilde{H}^1(\beta, \Lambda) = \widetilde{H}(\beta, \Lambda) \cap \widetilde{P}_1(\Lambda)$ and $\widetilde{J}^1(\beta, \Lambda) = \widetilde{J}(\beta, \Lambda) \cap \widetilde{P}_1(\Lambda)$, respectively.

We can set $H(\beta, \Lambda) = \widetilde{H}(\beta, \Lambda) \cap G$, $H^1(\beta, \Lambda) = \widetilde{H}^1(\beta, \Lambda) \cap G$, and similarly $J(\beta, \Lambda)$, $J^1(\beta, \Lambda)$.

Let θ be a skew simple character of $H^1(\beta, \Lambda)$, defined by [17, 3.2], that is, θ is the restriction of a simple character $\widetilde{\theta}$ of $\widetilde{H}(\beta, \Lambda)$ to $H(\beta, \Lambda)$, defined by [2, (3.2)], with $\widetilde{\theta}^{\gamma} = \widetilde{\theta}$. Then by [19, Proposition 3.5] there is a unique irreducible representation η of $J^1(\beta, \Lambda)$ such that η contains θ, and by [19, Definition 4.5] there is a β-extension κ of η to $J(\beta, \Lambda)$ relative to Λ^{M}. This definition is general. However for the underlying stratum $[\Lambda', n', 0, \beta]$ of $[\Lambda, n, 0, \beta]$ in Definition 1.1, it is proved by [13, Proposition 3.8] that the β-extension κ is intertwined by the whole $B^{\times} \cap G$. This property is analogous to that for $GL(N, F)$ of [2, (5.2.1)] and is stronger than that of [19, Corollary 4.6] in the general case (see the remarks below that corollary).

It follows from [13, 5.3] that for the integer $f = R/e$, there is a canonical isomorphism

$$J(\beta, \Lambda)/J^1(\beta, \Lambda) \simeq \begin{cases} GL(f, k_E)^m & \text{if } e \text{ is even}, \\ GL(f, k_E)^m \times U(f, k_{E_0}) & \text{if } e \text{ is odd}, \end{cases}$$

where $U(f, k_{E_0})$ is the unitary group of a non-degenerate k_E/k_{E_0}-skew-hermitian form of dimension f over k_E (cf. [19, 3.3 and Corollary 5.11]). Let τ_0 (resp. τ_1) be an irreducible cuspidal representation of $GL(f, k_E)$ (resp. $U(f, k_{E_0})$). We define an irreducible representation τ by $\tau = \tau_0^{\otimes m}$ if e is even, and by $\tau = \tau_0^{\otimes m} \otimes \tau_1$ if e is odd. Via this isomorphism, we lift τ to an irreducible representation, say again τ, of $J(\beta, \Lambda)$.

Form the β-extension κ above and this representation τ, we define a representation λ of $J(\beta, \Lambda)$ by

$$\lambda = \kappa \otimes \tau.$$

We call it a *simple type* (of positive level) in G attached to a good skew simple stratum $[\Lambda, n, 0, \beta]$ in A.

1.4. We assume that $[\Lambda, n, 0, \beta]$ is a good skew simple stratum in A. Let $E = F[\beta]$, and B the A-centralizer of β.

From the E-composition (1.1), we have a Levi subgroup $\widetilde{M}$ of $\widetilde{G}$ which is defined by the stabilizer of (1.1) in $\widetilde{G}$. We set $M = \widetilde{M} \cap G$. Then M is a Levi subgroup of G which is isomorphic to $\prod_{j=1}^{m} \mathrm{Aut}_F(W^{(j)})$ if e is even, and to $(\prod_{j=1}^{m} \mathrm{Aut}_F(W^{(j)})) \times G_0$ if e is odd, where G_0 is a unitary group of the same type as G. Let P be a parabolic subgroup of G such that elements of its unipotent radical, denoted by U, have upper triangular block form. Then $P = MU$.

By [13, Lemma 2.6], there is a canonical Witt basis $\mathcal{V}$ of $(V, \widetilde{h}_\beta)$ (cf. [2, (5.5.1)]). Let $\boldsymbol{W}$ be the affine Weyl group of $G_E = B^\times \cap G$ with respect to a maximal torus T_E defined by $\mathcal{V}$, and $\boldsymbol{W}_m$ the normalizer of $\mathfrak{b}_0(\Lambda) \cap \widetilde{M}$ in $\boldsymbol{W}$.

For an integer j $(1 \leqslant j \leqslant m)$, we obtain the Weyl element s_j of $\boldsymbol{W}_m$ by [19, 6.2], and define the involution σ_j on the factor $\mathrm{Aut}_F(W^{(j)})$ of M by $\sigma_j(g) = s_j g (s_j)^{-1}$ for $g \in \mathrm{Aut}_F(W^{(j)})$ following [19, 6.3]. This involution σ_j induces the one, denoted by $\overline{\sigma}_j$, on $GL(f, k_E)$. We say that the representation τ of $J(\beta, \Lambda)$ in 1.3 is *self-dual* if the factor τ_0 of $GL(f, k_E)$ satisfies $\tau_0 \simeq \tau_0 \circ \overline{\sigma}_j$.

By [2, (7.1.14)], the groups $H^1(\beta, \Lambda)$, $J^1(\beta, \Lambda)$, and $J(\beta, \Lambda)$ have Iwahori decompositions with respect to (M, P) (cf. [19, Corollary 5.10]), and so we have a compact open subgroup of G defined by $J_P = H^1(\beta, \Lambda)(J(\beta, \Lambda) \cap P)$. For the simple type λ in 1.3, we let λ_P be the natural representation of J_P on the space of $(J(\beta, \Lambda) \cap U)$-fixed vectors in the representation space of λ.

Let $\mathcal{H}(G, \lambda)$ and $\mathcal{H}(G, \lambda_P)$ be the Hecke algebras of $(J(\beta, \Lambda), \lambda)$ and (J_P, λ_P), respectively (cf. [2, Section 4]). We say that λ (resp. λ_P) is *self-dual* if its factor τ is self-dual. For our unitary group G, it follows easily from [19, 7.1] that if λ (resp. λ_P) is self-dual, functions of $\mathcal{H}(G, \lambda)$ (resp. $\mathcal{H}(G, \lambda_P)$) are all supported on $\boldsymbol{W}_m$, and that there is a canonical support-preserving isomorphism $\mathcal{H}(G, \lambda) \simeq \mathcal{H}(G, \lambda_P)$.

Theorem 1.4. *(See Theorem 6.6 of [13].) Let $[\Lambda, n, 0, \beta]$ be a good skew simple stratum in A, and λ_P the representation of J_P obtained from a simple type $(J(\beta, \Lambda), \lambda)$ in G attached to $[\Lambda, n, 0, \beta]$. Then there is an irreducible supercuspidal representation, π_M, of M such that (J_P, λ_P) is an $[M, \pi_M]_G$-type in G.*

1.5. Let $[\Lambda, n, 0, \beta]$ be a good skew simple stratum in A, land λ_P be the representation of J_P obtained from a simple type $(J(\beta, \Lambda), \lambda)$ in G attached to $[\Lambda, n, 0, \beta]$.

In order to study the Hecke algebra $\mathcal{H}(G, \lambda_P)$ in 1.4, we recall the structure of the compact open subgroups $\widetilde{H}^1(\beta, \Lambda)$, $\widetilde{J}^1(\beta, \Lambda)$, and $\widetilde{J}(\beta, \Lambda)$ of $\widetilde{G}$ obtained by [6].

Let $E = F[\beta]$, $e = e(\Lambda_{\mathfrak{o}_E})$, and B the A-centralizer of β. Let $m = \lfloor e/2 \rfloor$ as in 1.2. For the decomposition (1.1) of V as F-vector spaces, we rewrite $V_1 = W^{(-m)}$, $V_2 = W^{(-m+1)}$, $\ldots$, $V_e = W^{(m)}$, and $\Lambda_1 = \Lambda^{(-m)}$, $\Lambda_2 = \Lambda^{(-m+1)}$, $\ldots$, $\Lambda_e = \Lambda^{(m)}$, temporary. Then $V = \bigoplus_{i=1}^{e} V_i$ and $\Lambda(k) = \bigoplus_{i=1}^{e} \Lambda_i(k)$ $(k \in \mathbb{Z})$. Thus A is decomposed as follows

$$A = \coprod_{i,j} A^{ij}, \quad A^{ij} = \mathrm{Hom}_F(V_j, V_i),$$

where i, j run through $\{1, \ldots, e\}$. By Lemma 1.3, we can choose an $\mathfrak{o}_E$-basis $\mathcal{V}_i$ of Λ_i in V_i appropriately, for each i $(1 \leqslant i \leqslant e)$ (cf. [2, (1.1.7)]). We identify $\mathcal{V}_1 = \cdots = \mathcal{V}_e$, and so $V_1 = \cdots = V_e$. We write $W = V_i$ and $\Lambda_W = \Lambda_i$, for all i. Thus we can regard $A^{ij} = \mathrm{End}_F(W)$ $(1 \leqslant i, j \leqslant e)$.

By definition, we have $\widetilde{M} = (\prod_i A^{ii}) \cap \widetilde{G}$. We set $\mathbb{N} = \prod_{i<j} A^{ij}$, $\mathbb{N}_\ell = \prod_{i>j} A^{ij}$, $\widetilde{U} = 1 + \mathbb{N}$, and $\widetilde{U}_\ell = 1 + \mathbb{N}_\ell$. Let $\widetilde{P} = \widetilde{M}\widetilde{U}$ and $\widetilde{J}_{\widetilde{P}} = \widetilde{H}^1(\beta, \Lambda)(\widetilde{J}(\beta, \Lambda) \cap \widetilde{P})$ be the subgroup of $\widetilde{G}$ defined in [2, (7.2.17)] (cf. [19, 5.1]). Then we have $P = \widetilde{P} \cap G$ and $J_P = \widetilde{J}_{\widetilde{P}} \cap G$.

Proposition 1.5.

(i) *The group $\widetilde{J}_{\widetilde{P}}$ has also an Iwahori decomposition with respect to $(\widetilde{M}, \widetilde{P})$, which induces that of J_P in* 1.4, *and we have*

$$\widetilde{J}_{\widetilde{\mathrm{P}}} \cap \widetilde{U}_\ell = \widetilde{H}^1(\beta, \Lambda) \cap \widetilde{U}_\ell = 1 + \big(\mathfrak{H}^1(\beta, \Lambda) \cap \mathbb{N}_\ell\big),$$
$$\widetilde{J}_{\widetilde{\mathrm{P}}} \cap \widetilde{M} = \mathfrak{J}^0(\beta, \Lambda) \cap \widetilde{M},$$
$$\widetilde{J}_{\widetilde{\mathrm{P}}} \cap \widetilde{U} = \widetilde{J}^1(\beta, \Lambda) = 1 + \big(\mathfrak{J}^1(\beta, \Lambda) \cap \mathbb{N}\big).$$

(ii) *Assume that $e = e(\Lambda_{\mathfrak{o}_E})/2 \geqslant 2$. Then we have*
  (1) *for $i \in \{1, \ldots, e\}$, $\mathfrak{J}(\beta, \Lambda) \cap A^{ii} = \mathfrak{J}(\beta, \Lambda_W)$;*
  (2) *for $i, j \in \{1, \ldots, e\}$ with $i > j$,*

$$\mathfrak{J}(\beta, \Lambda) \cap A^{ij} = \begin{cases} \mathfrak{H}^1(\beta, \Lambda_W) & \text{if } 1 \leqslant i - j \leqslant \lfloor e/2 \rfloor, \\ \varpi_E \mathfrak{J}(\beta, \Lambda_W) & \text{if } \lfloor e/2 \rfloor + 1 \leqslant i - j \leqslant e - 1, \end{cases}$$
$$\mathfrak{J}(\beta, \Lambda) \cap A^{ji} = \begin{cases} \mathfrak{J}(\beta, \Lambda_W) & \text{if } 1 \leqslant i - j \leqslant \lceil e/2 \rceil - 1, \\ \varpi_E^{-1} \mathfrak{H}^1(\beta, \Lambda_W) & \text{if } \lceil e/2 \rceil \leqslant i - j \leqslant e - 1; \end{cases}$$

  (3) *for $i, j \in \{1, \ldots, e\}$ with $i > j$, we have*

$$\mathfrak{H}^1(\beta, \Lambda) \cap A^{ij} = \begin{cases} \mathfrak{H}^1(\beta, \Lambda_W) & \text{if } 1 \leqslant i - j \leqslant \lceil e/2 \rceil - 1, \\ \varpi_E \mathfrak{J}(\beta, \Lambda_W) & \text{if } \lceil e/2 \rceil \leqslant i - j \leqslant e - 1, \end{cases}$$
$$\mathfrak{J}^1(\beta, \Lambda) \cap A^{ji} = \begin{cases} \mathfrak{J}(\beta, \Lambda_W) & \text{if } 1 \leqslant i - j \leqslant \lceil e/2 \rceil - 1, \\ \varpi_E^{-1} \mathfrak{H}^1(\beta, \Lambda_W) & \text{if } \lceil e/2 \rceil \leqslant i - j \leqslant e - 1. \end{cases}$$

Proof. Part (i) follows from [2, (7.1.17)] and [19, 5.3], and part (ii) is a direct consequence of [6, Proposition 1 and Appendix]. □

2. An auxiliary unitary group $G_{\mathfrak{b}}$

In order to calculate the support of convolution products in the Hecke algebra $\mathcal{H}(G, \lambda_P)$, we define an auxiliary unitary group $G_{\mathfrak{b}}$.

In this section, we assume that $[\Lambda, n, 0, \beta]$ is a good skew simple stratum in $A = \mathrm{End}_F(V)$ with $e = e(\Lambda_{\mathfrak{o}_E})/2 \geqslant 2$. Let $E = F[\beta]$, $B = B_\beta$ A-centralizer of β, and $m = \lfloor e/2 \rfloor$. Associated with this β, we have the non-degenerate E/E_0-anti-hermitian form $\widetilde{h}_\beta$ defined in 1.1. By [7, 23.9], we may assume that the form $\widetilde{h}_\beta$ is hermitian, and so do we hereafter. Let $V = \bigoplus_{j=-m}^{m} W^{(j)}$ be the E-decomposition (1.1) of V, with $\dim_E(W^{(j)}) = f$, for all j $(-m \leqslant j \leqslant m)$. We choose a Witt basis $\mathcal{V}$ for $(V, \widetilde{h}_\beta)$ as follows: We denote by $\widetilde{h}_\beta^j$ the restriction of $\widetilde{h}_\beta$ to the subspace $W^{(-j)} + W^{(j)}$ for $j \geqslant 0$. For $j \in \{1, \ldots, m\}$, we choose an ordered $\mathfrak{o}_E$-basis $\mathcal{V}^{(j)} = \{v_1^j, \ldots, v_f^j\}$ for the lattice $\Lambda^{(j)}(0)$ such that $\Lambda^{(j)}(0) = \mathfrak{o}_E v_1^j \oplus \cdots \oplus \mathfrak{o}_E v_f^j$. We take the ordered basis $\mathcal{V}^{(-j)} = \{v_1^{-j}, \ldots, v_j^{-j}\}$ of $W^{(-j)}$ for which $\widetilde{h}_\beta^j(v_s^j, v_t^{-j}) = \delta_{s,t}$, where $\delta_{s,t}$ denotes the Kronecker delta. If e is odd, we choose a Witt E-basis $\mathcal{V}^{(0)}$ such that $\Lambda^{(0)}(k) = \bigoplus_v \mathfrak{p}_E^{n(k)} v$, for $n(k) \in \mathbb{Z}$, where v runs through $\mathcal{V}^{(0)}$. Here we note that f is even, since $R = ef$ is so (see Definition 1.1). Form $\mathcal{V} = \coprod_{j=-m}^{m} \mathcal{V}^{(j)}$.

We set $I = \{\pm 1, \ldots, \pm m\}$. Let $\widetilde{I} = I$ if e is even, and $\widetilde{I} = I \cup \{0\}$ if e is odd. Let $V_{\mathfrak{b}}$ be an E-vector space with a (formal) basis $\{e_i \mid i \in \widetilde{I}\}$, and define an E/E_0-hermitian form $h_{\mathfrak{b}}$ on $V_{\mathfrak{b}}$ by

$$h_{\mathfrak{b}}(x, y) = \begin{cases} \sum_{i=1}^{m} (\overline{x_i} y_{-i} + x_{-i} \overline{y_i}) & \text{if } e \text{ is even}, \\ \overline{x_0} y_0 + \sum_{i=1}^{m} (\overline{x_i} y_{-i} + x_{-i} \overline{y_i}) & \text{if } e \text{ is odd} \end{cases}$$

where $x = \sum_k x_k e_k \in V$, $x_k \in E$, $\overline{x_k}$ denotes the E/E_0-Galois conjugate of x_k, and similarly $y = \sum_k y_k e_k$. Then this form is clearly a non-degenerate E/E_0-hermitian form. Denote by $G_{\mathfrak{b}} = U(V_{\mathfrak{b}}, h_{\mathfrak{b}})$ the unitary group of $(V_{\mathfrak{b}}, h_{\mathfrak{b}})$ over E_0. Then $\{e_i \mid i \in \bar{I}\}$ is the same notation as the canonical basis of $V_{\mathfrak{b}}$ in [9, Section 10] and [20, 1.15].

From now on, we express elements of $B = \mathrm{End}_E(V)$ in matrix form relative to the basis $\mathcal{V}$, and do so for $\mathrm{End}_E(V_{\mathfrak{b}})$ relative to the basis $\{e_i \mid i \in \widetilde{I}\}$. To each $(g_{ij}) \in \mathrm{End}_E(V_{\mathfrak{b}})$ with $g_{ij} \in E$, we naturally associate a family of E-linear maps $v \mapsto g_{ij}v$ from $W^{(j)}$ to $W^{(i)}$, which is again denoted by $g_{ij}I_f$, that is, $(g_{ij}I_f) \in B = \mathrm{End}_E(V)$, where I_f denotes the identity matrix of rank f. For $g = (g_{ij}) \in \mathrm{End}_E(V_{\mathfrak{b}})$, write

$$\varphi(g) = g \otimes I_f = (g_{ij}I_f) \in B = \mathrm{End}_E(V).$$

So we get an injection

$$\varphi : \mathrm{End}_E(V_{\mathfrak{b}}) \to B = \mathrm{End}_E(V). \tag{2.1}$$

For the hermitian space $(V, \widetilde{h}_\beta)$ above, $G_E = B^\times \cap G$ is naturally isomorphic to the unitary group $U(V, \widetilde{h}_\beta)$. Via this isomorphism, we identify $G_E = B^\times \cap G = U(V, \widetilde{h}_\beta)$.

Lemma 2.1. *The injective map φ of* (2.1) *induces an injective homomorphism*

$$\varphi : G_{\mathfrak{b}} = U(V_{\mathfrak{b}}, h_{\mathfrak{b}}) \to G_E = U(V, \widetilde{h}_\beta).$$

Proof. This follows immediately by definition (cf. [20, 1.15]). $\square$

We recall the structure of the unitary group $G_{\mathfrak{b}}$ by [9, Section 10], [20, 1.15], [15, Section 9], and [13, 5.1]:

We define a subgroup $S_{\mathfrak{b}}$ of $G_{\mathfrak{b}}$ which consists of the elements $\mathrm{Diag}(d_{-m}, \ldots, d_m)$ with $d_i \in E_0^\times$, and with $d_{-i}d_i = 1$ $(i \in I)$, if e is even, and with $d_{-i}d_i = d_0 = 1$ $(i \in I)$, if e is odd. Denote by $\boldsymbol{G}_{\mathfrak{b}}$, $\boldsymbol{S}_{\mathfrak{b}}$ the connected algebraic group generated by $G_{\mathfrak{B}}$, $S_{\mathfrak{b}}$ over E_0, respectively. Let $\boldsymbol{Z}_{\mathfrak{b}}$ be the centralizer of $\boldsymbol{S}_{\mathfrak{b}}$ in $\boldsymbol{G}_{\mathfrak{b}}$, and $Z_{\mathfrak{b}}$ the group of E_0-rational points in $\boldsymbol{Z}_{\mathfrak{b}}$.

For each i, we define a character $a_i : \boldsymbol{S}_{\mathfrak{b}} \to \boldsymbol{GL}_1$ by $a_i(\mathrm{Diag}(d_{-m}, \ldots, d_m)) = d_{-i}$, where $\boldsymbol{GL}_1$ denotes the multiplicative algebraic group defined over E_0. Then the character group, $X(\boldsymbol{S}_{\mathfrak{b}})$, of $\boldsymbol{S}_{\mathfrak{b}}$ is a free $\mathbb{Z}$-module generated by $a_1, \ldots, a_r$. In $X(\boldsymbol{S}_{\mathfrak{b}})$, put $a_{ij} = a_i + a_j$ and $a_{-i} = -a_i$. We define a subset, $\Phi_{\mathfrak{b}}$, of $X(\boldsymbol{S}_{\mathfrak{b}})$ by

$$\Phi_{\mathfrak{b}} = \begin{cases} \{a_{ij} \mid i, j \in I, i \neq \pm j\} \cup \{2a_i \mid i \in I\} & \text{if } e \text{ is even,} \\ \{a_{ij} \mid i, j \in I, i \neq \pm j\} \cup \{a_i, 2a_i \mid i \in I\} & \text{if } e \text{ is odd.} \end{cases}$$

Then this is the root system of $(\boldsymbol{G}_{\mathfrak{b}}, \boldsymbol{S}_{\mathfrak{b}})$.

For each $\alpha \in \Phi_{\mathfrak{b}}$, we define unipotent elements $x_\alpha(t) = 1 + (g_{k\ell})$ and $x_\alpha(t, u) = 1 + (g_{k\ell})$, $t, u \in E$, of $G_{\mathfrak{b}}$ by

(1) if $\alpha = a_{ij}$, for $t \in E$, $g_{-j,i} = -\bar{t}$, $g_{-i,j} = t$ and all other $g_{k\ell} = 0$,
(2) if $\alpha = 2a_i$, for $t \in E$ with $t + \bar{t} = 0$, $g_{-i,i} = t$ and all other $g_{k\ell} = 0$,
(3) if e is odd and $\alpha = a_i$, for $t, u \in E$ with $t\bar{t} + u + \bar{u} = 0$, $g_{-i,0} = -\bar{t}$, $g_{-i,i} = u$, $g_{0,i} = t$, and all other $g_{k\ell} = 0$.

Let $\Delta_{\mathfrak{b}} = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ of $\Phi_{\mathfrak{b}}$ be a basis of $\Phi_{\mathfrak{b}}$ such that the corresponding root subgroups are upper triangular unipotent. Since E/E_0 is unramified, it follows from [20, 1.15] that the affine root system of $G_{\mathfrak{b}}$ is given by $\widetilde{\Phi}_{\mathfrak{b}} = \{\alpha + \gamma \mid \alpha \in \Phi_{\mathfrak{b}}, \gamma \in \mathbb{Z}\}$. Let α_0 be the maximal root relative to $\Delta_{\mathfrak{b}}$. Then the set $\{\widetilde{\alpha}_0 = 1 - \alpha_0, \widetilde{\alpha}_i = \alpha_i \text{ for } i \in \{1, \ldots, m\}\}$ form a basis, say $\widetilde{\Delta}_{\mathfrak{b}}$, of $\widetilde{\Phi}_{\mathfrak{b}}$.

For each $\widetilde{\alpha}_i \in \widetilde{\Delta}_{\mathfrak{b}}$, let $s_i = s_{\widetilde{\alpha}_i}$ be the affine reflection associated with the affine root $\widetilde{\alpha}_i$. Then $G_{\mathfrak{b}}$ has an affine Weyl group $\boldsymbol{W}_{\mathfrak{b}}$ with a set of generators $\{s_0, s_1, \ldots, s_m\}$.

For the following argument, refer to [15, Section 9]. Let $X(\boldsymbol{Z}_\mathfrak{b}) = \mathrm{Hom}(\boldsymbol{Z}_\mathfrak{b}, \mathbb{G}_m)$ be the character group of $\boldsymbol{Z}_\mathfrak{b}$, $\widehat{X}(\boldsymbol{Z}_\mathfrak{b}) = \mathrm{Hom}(X(\boldsymbol{Z}_\mathfrak{b}), \mathbb{Z})$, and likewise for $\widehat{X}(\boldsymbol{S}_\mathfrak{b})$. Then, since E/E_0 is unramified, we may identify $X(\boldsymbol{Z}_\mathfrak{b}) = X(\boldsymbol{S}_\mathfrak{b}) = \mathbb{Z}^m$. Denote by L this $\mathbb{Z}$-module. Let $\boldsymbol{D}_\mathfrak{b}$ be the subgroup of $\boldsymbol{Z}_\mathfrak{b}$ whose coefficients consist of elements of form ϖ_E^n for $n \in \mathbb{Z}$. Then we have $\boldsymbol{D}_\mathfrak{b} = \{\varpi_E^{\boldsymbol{n}} \mid \boldsymbol{n} \in L = \mathbb{Z}^m\}$, where for $\boldsymbol{n} = (n_1, \ldots, n_m)$, $\varpi_E^{\boldsymbol{n}} = \mathrm{Diag}(\varpi_E^{n_m}, \ldots, \varpi_E^{n_1}, \varpi_E^{-n_1}, \ldots, \varpi_E^{-n_m})$ if e is even, and likewise for odd e.

Let $\langle\, , \rangle$ be the canonical pairing of $X(\boldsymbol{Z}_\mathfrak{b}) \times \widehat{X}(\boldsymbol{Z}_\mathfrak{b})$. Then, for $a_i \in X(\boldsymbol{Z}_\mathfrak{b}) = X(\boldsymbol{S}_\mathfrak{b})$, we have

$$a_i\big(\varpi_E^{\boldsymbol{n}}\big) = \varpi_E^{n_i} \in L = \widehat{X}(\boldsymbol{Z}_\mathfrak{b}) = \widehat{X}(\boldsymbol{S}_\mathfrak{b}).$$

Let $\boldsymbol{N}_\mathfrak{b}$ be the normalizer of $\boldsymbol{S}_\mathfrak{b}$ in $\boldsymbol{G}_\mathfrak{b}$, and denote by $N_\mathfrak{b}$ the group of E_0-rational points in $\boldsymbol{N}_\mathfrak{b}$. Let $Z_\mathfrak{b}^0 = \{\mathrm{Diag}(d_{-m}, \ldots, d_m) \in Z_\mathfrak{b} \mid d_i \in \mathfrak{o}_E^\times\}$. Then, since the derived subgroup of $\boldsymbol{G}_\mathfrak{b}$ is simply-connected, $\boldsymbol{W}_\mathfrak{b}$ is isomorphic to the quotient group $N_\mathfrak{b}/Z_\mathfrak{b}^0$ (cf. [20, 3.1]). Denote by $\boldsymbol{W}_{\mathfrak{b},0}$ the subgroup of $\boldsymbol{W}_\mathfrak{b}$ generated by $\{s_1, \ldots, s_r\}$. Then this is a finite Weyl group, and we have $\boldsymbol{W}_\mathfrak{b} = \boldsymbol{D}_\mathfrak{b} \rtimes \boldsymbol{W}_{\mathfrak{b},0}$ (semi-direct).

3. The group $G_E = B^\times \cap G$

We retain notation and assumptions in the previous Section 4.

Let $\boldsymbol{W}$ be the affine Weyl group of $G_E = B^\times \cap G = U(V, \widetilde{h}_\beta)$ associated with the basis $\mathcal{V}$ defined in Section 2, and $\boldsymbol{W}_m$ the normalizer of $\mathfrak{b}_0 \cap \widetilde{M}$ in $\boldsymbol{W}$ as in 1.4. Then, via the injective homomorphism φ in Lemma 2.1, we obtain $\boldsymbol{W}_\mathfrak{b} \simeq \boldsymbol{W}_m$. We set

$$\boldsymbol{D}_m = \varphi(\boldsymbol{D}_\mathfrak{b}) = \boldsymbol{D}_\mathfrak{b} \otimes I_f, \qquad \boldsymbol{W}_{m,0} = \varphi(\boldsymbol{W}_{\mathfrak{b},0}).$$

Then we have $\boldsymbol{W}_m = \boldsymbol{D}_m \rtimes \boldsymbol{W}_{m,0}$, and we may identify

$$\boldsymbol{D}_\mathfrak{b} = \boldsymbol{D}_m, \qquad \boldsymbol{W}_{\mathfrak{b},0} = \boldsymbol{W}_{m,0}, \qquad \boldsymbol{W}_\mathfrak{b} = \boldsymbol{W}_m.$$

Moreover, we denote again by $N_\mathfrak{b}$ the image $\varphi(N_\mathfrak{b})$ in G_E.

From the E-decomposition (1.1) of V, the F-algebra $A = \mathrm{End}_F(V)$ is decomposed into

$$A = \coprod_{i,j} \mathrm{Hom}_F\big(W^{(j)}, W^{(i)}\big),$$

where i, j run through $\widetilde{I}$. Put $A_{ij} = \mathrm{Hom}_F(W^{(j)}, W^{(i)})$.

Remark 3.1. For the elements $s_i \in \boldsymbol{W}_\mathfrak{b} = \boldsymbol{W}_m$ for $0 \leqslant i \leqslant m$ in the previous section, denote by $n_s \in N_\mathfrak{b} \subset G_E$ representatives for $s = s_i$. We choose s_m^ϖ and s_{m-i} $(0 \leqslant i \leqslant m-1)$ defined in [19, 6.2] as the representatives n_{s_0} and n_{s_i} $(1 \leqslant i \leqslant m)$, respectively.

Let $i, j \in \widetilde{I}$. For a block matrix $T \in \mathrm{Hom}_F(W^{(i)}, W^{(-j)})$, we have

$$h(Tv, w) = h\big(v, \sigma(T)w\big), \quad v \in W^{(i)},\ w \in W^{(j)},$$

and $\sigma(T) \in \mathrm{Hom}_F(W^{(j)}, W^{(-i)})$.

Definition 3.2. Let $\Phi_\mathfrak{b}$ be the root system of $G_\mathfrak{b}$ defined in Section 2. For each $\alpha \in \Phi_\mathfrak{b}$, we define unipotent elements of $\widetilde{G} = A^\times$ in the block form as follows:

(1) For $\alpha = a_{ij} \in \Phi_{\mathfrak{b}}$, $(T, T') \in A_{-i,j} \times A_{-j,i}$, set $\widetilde{x}_{a_{ij}}(T, T') = 1 + y$, where $y_{-j,i} = T'$, $y_{-i,j} = T$ and all other block $y_{k\ell} = 0$.
(2) For $\alpha = 2a_i \in \Phi_{\mathfrak{b}}$, $T \in A_{-i,i}$, set $\widetilde{x}_{2a_i}(T) = 1 + y$, where $y_{-i,i} = T$ and all other block $y_{k\ell} = 0$.
(3) For $\alpha = a_i \in \Phi_{\mathfrak{b}}$, $(T, T', T'') \in A_{i0} \times A_{i,-i} \times A_{0,-i}$, set $\widetilde{x}_{a_i}(T, T', T'') = 1 + y$, where $y_{0,-i} = T''$, $y_{i,-i} = T'$, $y_{i0} = T$ and all other block $y_{k\ell} = 0$.

Lemma 3.3. *We have*

(1) $\widetilde{x}_{a_{ij}}(T, T') \in G \Leftrightarrow \sigma(T) + T' = 0$;
(2) $\widetilde{x}_{2a_i}(T) \in G \Leftrightarrow \sigma(T) + T = 0$;
(3) $\widetilde{x}_{a_i}(T, T', T'') \in G \Leftrightarrow \sigma(T) + T'' = 0$ *and* $T' + \sigma(T') + \sigma(T'')T'' = 0$.

Proof. For (1), let $x = \widetilde{x}_{a_{ij}}(T, T')$. Then $x \in G$ if and only if $h(xv, xw) = h(v, w)$ for all $v, w \in V$. Let $v = \sum_k v_k$, $w = \sum_k w_k$, $v_k, w_k \in W^{(k)}$ $(k \in \widetilde{I})$. By a simple calculation, we see that the above condition is equivalent to the following: For any $v_i, w_i \in W^{(i)}$ and any $v_j, w_j \in W^{(j)}$,

$$h\big(v_i, \big(\sigma(T) + T'\big)w_j\big) + h\big(v_j, \big(\sigma\big(T'\big) + T\big)w_i\big) = 0.$$

Thus this is equivalent to $\sigma(T) + T' = 0$, which complete the proof of (1). The condition (2) follows similarly.

For (3), let $x = \widetilde{x}_{a_i}(T, T', T'')$. For $v = \sum_k v_k$, $w = \sum_k w_k \in V$ of (1), $x \in G$ if and only if, for any $v_0, w_0 \in W^{(0)}$ and any $v_{-i}, w_{-i} \in W^{(-i)}$, we have

$$h\big(v_0, \big(\sigma(T) + T''\big)w_{-i}\big) + h\big(v_{-i}, \big(\sigma\big(T''\big) + T\big)w_0\big) + h\big(v_{-i}, \big(T' + \sigma\big(T'\big) + \sigma\big(T''\big)T''\big)w_{-i}\big) = 0.$$

This is clearly equivalent to $\sigma(T) + T'' = 0$ and $T' + \sigma(T') + \sigma(T'')T'' = 0$. The proof is completed. □

Hereinafter, if $\widetilde{x}_{a_{ij}}(T, T'), \widetilde{x}_{a_i}(T, T', T'') \in G$, then we set

$$\widetilde{x}_{a_{ij}}(T) = \widetilde{x}_{a_{ij}}\big(T, -\sigma(T)\big),$$
$$\widetilde{x}_{a_i}\big(T, T'\big) = \widetilde{x}_{a_i}\big(T, T', -\sigma(T)\big).$$

An element w of $\boldsymbol{W}_{\mathfrak{b}} = \boldsymbol{W}_m$ can be uniquely written in the form $w = dw_0$, where $d = \varpi_E^{\boldsymbol{n}} \in \boldsymbol{D}_{\mathfrak{b}} = \boldsymbol{D}_m$, $\boldsymbol{n} \in L = \mathbb{Z}^m$, and $w_0 \in \boldsymbol{W}_{\mathfrak{b},0} = \boldsymbol{W}_{m,0}$. Moreover, we have $d = \varpi_E^{\boldsymbol{n}} \otimes I_f$ in G_E, and we may set the representative n_{w_0} of w_0 in $N_{\mathfrak{b}} \subset G_E$ so that its block coefficients consist of $\pm I_f$ and 0. Thus

$$dn_{w_0} = \big(\varpi_E^{\boldsymbol{n}} \otimes I_f\big)n_{w_0}$$

is a representative of $w = dw_0 \in \boldsymbol{W}_{\mathfrak{b}} = \boldsymbol{W}_m$ in $N_{\mathfrak{b}} \subset G_E$.

Proposition 3.4. *Let $w = dw_0 \in \boldsymbol{W}_{\mathfrak{b}}$ with $d = \varpi_E^{\boldsymbol{n}}$, $\boldsymbol{n} \in L = \mathbb{Z}^m$, $w_0 \in \boldsymbol{W}_{\mathfrak{b},0}$, and n_{w_0} be as above. Put $n_w = dn_{w_0}$. Then, for $\widetilde{x}_\alpha(T), \widetilde{x}_\alpha(T, T') \in G$ and $\alpha \in \Phi_{\mathfrak{b}}$,*

(1) *For $\alpha = a_{ij}, 2a_i$ $(i \in I)$,*

$$n_w \widetilde{x}_\alpha(T) n_w^{-1} = \widetilde{x}_{w_0(\alpha)}\big(\varpi_E^{\langle w_0(\alpha), \boldsymbol{n}\rangle} T_1\big);$$

where $T_1 = \pm\varpi_E^b T \varpi_E^{-b}$ for some integer b.

(3) *For* $\alpha = a_i$ $(i \in I)$,

$$n_w \widetilde{x}_\alpha\big(T, T'\big) n_w^{-1} = \widetilde{x}_{w_0(\alpha)}\big(\varpi_E^{\langle w_0(\alpha), \boldsymbol{n}\rangle} T_1, \varpi_E^{\langle w_0(2\alpha), \boldsymbol{n}\rangle} T_1'\big)$$

where $T_1 = \pm \varpi_E^b T \varpi_E^{-b}$ *and* $T_1' = \pm \varpi_E^{b'} T \varpi_E^{-b'}$ *for some integers* b, b'.

Proof. For (1), let $\alpha = a_{ij} = a_i + a_j \in \Phi_{\mathfrak{b}}$ $(i \neq \pm j)$, and $T \in A_{-i,j}$. Then

$$\begin{aligned} n_w \widetilde{x}_{a_{ij}}(T) n_w^{-1} &= \widetilde{x}_{w_0(a_{ij})}\big(\pm \varpi_E^{\langle w_0(a_i), \boldsymbol{n}\rangle} T \varpi_E^{\langle w_0(a_j), \boldsymbol{n}\rangle}\big) \\ &= \widetilde{x}_{w_0(a_{ij})}\big(\varpi_E^{\langle w_0(a_{ij}), \boldsymbol{n}\rangle} T_1\big), \end{aligned}$$

where $T_1 = \pm \varpi_E^b T \varpi_E^{-b}$ with $b = \langle -w_0(a_j), \boldsymbol{n}\rangle$.

For $\alpha = 2a_i = a_i + a_i$, similarly.

(2) For $\alpha = a_i$, we remark that $w_0(\alpha) \in \{a_i \mid i \in I\}$, since n_{w_0} acts as the multiplication by ± 1 on V_0. Similarly, we have

$$\begin{aligned} n_w \widetilde{x}_{a_i}\big(T, T'\big) n_w^{-1} &= \widetilde{x}_{w_0(a_i)}\big(\epsilon_1 T \varpi_E^{\langle w_0(a_i), \boldsymbol{n}\rangle}, \epsilon_2 \varpi_E^{\langle w_0(a_i), \boldsymbol{n}\rangle} T' \varpi_E^{\langle w_0(a_i), \boldsymbol{n}\rangle}\big) \\ &= \widetilde{x}_{w_0(a_i)}\big(\varpi_E^{\langle w_0(a_i), \boldsymbol{n}\rangle} T_1, \varpi_E^{\langle w_0(2a_i), \boldsymbol{n}\rangle} T_1'\big) \end{aligned}$$

where $\epsilon_i = \pm 1$ $(i = 1, 2)$, $T_1 = \epsilon_1 \varpi_E^b T \varpi_E^{-b}$ with $b = \langle w_0(-a_i), \boldsymbol{n}\rangle$, and T_1' similarly. The proof is completed. $\square$

Since $\boldsymbol{W}_{\mathfrak{b}}$ and $\{s_0, s_1, \ldots, s_r\}$ form a Coxeter system, as was seen in Section 2, the affine Weyl group $\boldsymbol{W}_{\mathfrak{b}}$ has the length function $\ell_{\mathfrak{b}}$.

Proposition 3.5. *Suppose that* $w = d w_0 \in \boldsymbol{W}_{\mathfrak{b}}$, $d = \varpi_E^{\boldsymbol{n}}$, $\boldsymbol{n} \in L = \mathbb{Z}^m$, *and* $w_0 \in \boldsymbol{W}_{\mathfrak{b},0}$. *Let* $\ell_{\mathfrak{b}}(w s_0) > \ell_{\mathfrak{b}}(w)$. *Then*

$$\begin{cases} \varpi_E^{\langle w_0(-\alpha_0), \boldsymbol{n}\rangle} \in \mathfrak{o}_E & \text{if } w_0(-\alpha_0) < 0, \\ \varpi_E^{\langle w_0(-\alpha_0), \boldsymbol{n}\rangle} \in \mathfrak{p}_E^{-1} & \text{if } w_0(-\alpha_0) > 0. \end{cases}$$

Proof. (Cf. [12, Proposition 1.28].) It follows from [10, 1.6 Proposition] that

$$\ell_{\mathfrak{b}}(w s_i) > \ell_{\mathfrak{b}}(w) \quad \Longleftrightarrow \quad w(\widetilde{\alpha}_i) > 0.$$

Let $i = 0$. Since $\widetilde{\alpha}_0 = 1 - \alpha_0$, we have

$$w(\widetilde{\alpha}_0) = w(1 - \alpha_0) = 1 + \big\langle w_0(-\alpha_0), \boldsymbol{n}\big\rangle + w_0(-\alpha_0).$$

Thus, from [10, 1.4], $w(\widetilde{\alpha}_0) > 0$ if and only if

$$\begin{cases} \text{“}w_0(-\alpha_0) < 0 \text{ and } 1 + \big\langle w_0(-\alpha_0), \boldsymbol{n}\big\rangle > 0\text{” or,} \\ \text{“}w_0(-\alpha_0) > 0 \text{ and } 1 + \big\langle w_0(-\alpha_0), \boldsymbol{n}\big\rangle \geqslant 0\text{”.} \end{cases}$$

This shows the proposition. $\square$

4. The Hecke algebra $\mathcal{H}(G, \lambda_P)$

In this section, assume that $[\Lambda, n, 0, \beta]$ is a good skew simple stratum in $A = \mathrm{End}_F(V)$ with $2e = e(\Lambda_{\mathfrak{o}_E}) \geqslant 4$. Let (J_P, λ_P) be a self-dual simple type in G attached to $[\Lambda, n, 0, \beta]$, and $\boldsymbol{W}_m$ the affine Weyl group with the basis $\{s_0, s_1, \ldots, s_m\}$ which is the support of the Hecke algebra $\mathcal{H}(G, \lambda_P)$ (see 1.4).

Proposition 4.1. *The Hecke algebra $\mathcal{H}(G, \lambda_P)$ has a basis T_w, for $w \in \boldsymbol{W}_m$, as a $\mathbb{C}$-vector space such that $T_{s_0}, T_{s_1}, \ldots, T_{s_m}$ satisfy the following quadratic relations:*

$$T_{s_i}^2 = a_i T_{s_i} + b_i 1, \quad a_i, b_i \in \mathbb{C},\ b_i \neq 0 \tag{4.1}$$

with $(a_i, b_i) = (q_E^f - 1, q_E^f)$ for $1 \leqslant i \leqslant m-1$. Note that the values of the parameters (a_0, b_0) and (a_m, b_m) are not determined.

Proof. By [13, Proposition 6.2], the Hecke algebra $\mathcal{H}(G, \lambda_P)$ has a basis T_w, for $w \in \boldsymbol{W}_m$, as a $\mathbb{C}$-vector space.

For the following facts, refer to [19, 7.2.2]. For $t = 0, 1$, we have the self-dual $\mathfrak{o}_E$-lattice sequence $\mathfrak{M}_t$ in V and support-preserving injective algebra maps, ϕ_t,

$$\mathcal{H}\big(P(\mathfrak{M}_{t,\mathfrak{o}_E}), \rho_t\big) \to \mathcal{H}(G, \lambda_P).$$

We remark $\mathfrak{M}_1 = \Lambda^{\mathrm{M}}$, as in the proof of Lemma 1.3. By the choice of $\mathfrak{M}_t$, there exist $T_0 \in \mathcal{H}(P(\mathfrak{M}_{0,\mathfrak{o}_E}), \rho_0)$ and $T_1, \ldots, T_m \in \mathcal{H}(P(\mathfrak{M}_{1,\mathfrak{o}_E}), \rho_1)$ such that $\phi_0(T_0) = T_{s_0}$ and $\phi_1(T_i) = T_{s_i}$ for $1 \leqslant i \leqslant m$, respectively. Thus [14, Theorem 7.12] shows the quadratic relations (4.1) for the T_s's in the assertion.

We prove $(a_i, b_i) = (q_E^f - 1, q_E^f)$ for $1 \leqslant i \leqslant m-1$. We may assume $m = 2$ (then $e = 4$ or $e = 5$) and it is enough to see $(a_1, b_1) = (q_E^f - 1, q_E^f)$ for T_1. Set $\overline{G} = GL(2f, k_E)$, $\overline{M} = GL(f, k_E) \times GL(f, k_E)$, and $D = \tau_0 \otimes \tau_0$. We denote by $\overline{P}$ a maximal parabolic subgroup of $\overline{G}$ with Levi component $\overline{M}$. By definition, $P(\Lambda^{\mathrm{M}}_{\mathfrak{o}_E})/P_1(\Lambda^{\mathrm{M}}_{\mathfrak{o}_E})$ is isomorphic to a unitary group $U(ef, k_{E_0})$ (cf. [19, 3.3]). The group $\overline{G}$ is naturally embedded into $P(\Lambda^{\mathrm{M}}_{\mathfrak{o}_E})/P_1(\Lambda^{\mathrm{M}}_{\mathfrak{o}_E})$, and $\rho_1 = D$ or $\rho_1 = D \otimes \tau_1$. Thus we see that the subalgebra of $\mathcal{H}(P(\Lambda^{\mathrm{M}}_{\mathfrak{o}_E}), \rho_1)$ generated by T_1 is isomorphic to $E(D) = \mathrm{Ind}_{\overline{P}}^{\overline{G}}(D)$, and hence $(a_1, b_1) = (q_E^f - 1, q_E^f)$ follows from [11, (4.15)] (cf. [6, Proposition 6]). The proof is completed. $\square$

Let $[\Lambda^{\mathrm{M}}, n_{\mathrm{M}}, 0, \beta]$ be a skew semi-simple stratum in A with Λ^{M} in Lemma 3.1, and the β-extension κ in 1.3 be compatible with some β-extension κ_{M} of $J(\beta, \Lambda^{\mathrm{M}})$ in the sense of [19, Definition 4.5]. Write $J^t_{\mathrm{M}} = J^t(\beta, \Lambda^{\mathrm{M}})$, for $t = 0, 1$. For the representation $\lambda = \kappa \otimes \tau$ of $J(\beta, \Lambda)$ in 1.3, we set $\lambda' = (\kappa_{\mathrm{M}}|_{P(\Lambda_{\mathfrak{o}_E})J^1_{\mathrm{M}}}) \otimes \tau$, a representation of $P(\Lambda_{\mathfrak{o}_E})J^1_{\mathrm{M}}$ (cf. [2, (5.6.1)]). By [19, Proposition 7.1], there is a canonical algebra isomorphism

$$\mathcal{H}\big(G, \lambda'\big) \simeq \mathcal{H}(G, \lambda_P) \tag{4.2}$$

which is support-preserving.

Let $\ell(w)$ be the length of an element w of $\boldsymbol{W}_m$. We remark that $\ell(w) = \ell_{\mathfrak{b}}(w)$ via the identification $\boldsymbol{W}_m = \boldsymbol{W}_{\mathfrak{b}}$.

Proposition 4.2. *Let $w \in \boldsymbol{W}_m$, $s = s_i$, for $i \in \{0, 1, \ldots, m\}$, and n_w, n_s be these representatives in $N_{\mathfrak{b}}$ as in Remark* 3.1. *Let T_w be a function in $\mathcal{H}(G, \lambda_P)$ supported on $J_P n_w J_P$. If $\ell(ws) > \ell(w)$, then $T_w * T_s$ is supported on $J_P n_w n_s J_P$.*

Proof. Assume that $\ell(ws) > \ell(w)$. We will prove

$$J_P n_w J_P n_s J_P = J_P n_w n_s J_P. \tag{4.3}$$

We have a canonical bijection

$$J_P \backslash J_P n_s J_P \simeq J_P / \big(J_P \cap n_s J_P n_s^{-1}\big).$$

Denote by Θ the complete system of representatives of the right-hand side. Then we have $J_P n_s J_P = \bigcup_{k\in\Theta} k n_s J_P$, and

$$J_P n_w J_P n_s J_P = \bigcup_{k\in\Theta} J_P n_w k n_w^{-1} n_w n_s J_P.$$

For each $k \in \Theta$, it is enough to show $n_w k n_w^{-1} \in J_P$ (cf. [12, Proposition 2.8]). This method is valid only for $s = s_0$.

(I) Assume $s = s_0$. Write $n_0 = n_{s_0}$ in Remark 3.1. We first remark that the action on the blocks A^{ij} of $A = \coprod_{i,j} A^{ij}$ in 1.5 by the conjugation $x \mapsto n_0 x n_0^{-1}$ is described as follows: $A^{i1} \to A^{ie}$: $x \mapsto x\varpi_E^{-1}$ $(1 < i < e)$, $A^{e1} \to A^{1e}$: $x \mapsto -\varpi_E^{-1} x \varpi_E^{-1}$, $A^{ej} \to A^{1j}$: $x \mapsto \varpi_E^{-1} x$ $(1 < j < e)$, and it is trivial on the other A^{ij}, and that $\mathfrak{J}^0(\beta, \Lambda_W) \subset \varpi_E^{-1}\mathfrak{H}^1(\beta, \Lambda_W)$ by [5, 2.1]. From this remark and Proposition 1.5, by using elementary row and column operations, we can choose $k \in \Theta$ in the following form: If e is even, $k = \widetilde{x}_{-\alpha_0}(T)$ with $T \in \varpi_E \mathfrak{J}^0(\beta, \Lambda_W)/\varpi_E \mathfrak{H}^1(\beta, \Lambda_W)$, and if e is odd, $k = \widetilde{x}_{-\alpha_0/2}(T, T')$ with $T \in \mathfrak{H}^1(\beta, \Lambda_W)/\varpi_E \mathfrak{J}^0(\beta, \Lambda_W)$, $T' \in \varpi_E \mathfrak{J}^0(\beta, \Lambda_W)/\varpi_E \mathfrak{H}^1(\beta, \Lambda_W)$. Here we remark $k \in J_P \cap U_\ell$.

We now assume that e is odd. For $w \in \boldsymbol{W}_m = \boldsymbol{W}_{\mathfrak{b}}$, let $w = d w_0$, with $d = \varpi_E^{\boldsymbol{n}}$, $\boldsymbol{n} \in L = \mathbb{Z}^m$, and with $w_0 \in \boldsymbol{W}_{m,0}$. Then, by Proposition 3.4, we have

$$n_w k n_w^{-1} = \widetilde{x}_{w_0(-\alpha_0/2)}\big(\varpi_E^{\langle w_0(-\alpha_0/2), \boldsymbol{n}\rangle} T_1, \varpi_E^{\langle w_0(-\alpha_0), \boldsymbol{n}\rangle} T_1'\big).$$

By definition, $T_1 \in \mathfrak{H}^1(\beta, \Lambda_W)$ and $T_1' \in \varpi_E \mathfrak{J}^0(\beta, \Lambda_W)$.

(1) Let $w_0(-\alpha_0/2) < 0$. Then $w_0(-\alpha_0) < 0$, and by Proposition 3.5, we have $\varpi_E^{\langle w_0(-\alpha_0), \boldsymbol{n}\rangle} \in \mathfrak{o}_E$ and $\varpi_E^{\langle w_0(-\alpha_0/2), \boldsymbol{n}\rangle} \in \mathfrak{o}_E$. Thus $\varpi_E^{\langle w_0(-\alpha_0), \boldsymbol{n}\rangle} T_1 \in \mathfrak{H}^1(\beta, \Lambda_W)$ and $\varpi_E^{\langle w_0(-\alpha_0/2), \boldsymbol{n}\rangle} T_1' \in \varpi_E \mathfrak{J}^0(\beta, \Lambda_W)$. Hence, from the block form of J_P in Proposition 1.5, we can see that $n_w k n_w^{-1} \in J_P$.

(2) Next, let $w_0(-\alpha_0/2) > 0$. Then $w_0(-\alpha_0) > 0$. Again by Proposition 3.5, we have $\varpi_E^{\langle w_0(-\alpha_0), \boldsymbol{n}\rangle} \in \mathfrak{p}_E^{-1}$. So $\varpi_E^{\langle w_0(-\alpha_0/2), \boldsymbol{n}\rangle} \in \mathfrak{o}_E$. Thus we have

$$\varpi_E^{\langle w_0(-\alpha_0/2), \boldsymbol{n}\rangle} T_1 \in \mathfrak{H}^1(\beta, \Lambda_W) \quad \text{and} \quad \varpi_E^{\langle w_0(-\alpha_0), \boldsymbol{n}\rangle} T_1' \in \mathfrak{J}^0(\beta, \Lambda_W).$$

Similarly Proposition 1.5 shows $n_w k n_w^{-1} \in J_P$.

The assertion in the case of even e is proved in the calculation of the term T' above.

(II) Let $s = s_i$, $i \in \{1, \dots, m\}$. In this case, we can never take a representative k in Θ so that $k \in J_P \cap U_\ell$ or $k \in J_P \cap U$, as above. So we cannot use the method of (I). However, fortunately, we can generalize the method of [2, (5.6.11)] for $GL(N, F)$ to G. Via the isomorphism (4.2), we pass to the representation $\lambda' = (\kappa_M | P(\Lambda_{\mathfrak{o}_E}) J_{\mathsf{M}}^1) \otimes \tau$ of $P(\Lambda_{\mathfrak{o}_E}) J_{\mathsf{M}}^1$. Thus, in order to prove (4.3), it is enough to prove

$$P(\Lambda_{\mathfrak{o}_E}) J_{\mathsf{M}}^1 n_w P(\Lambda_{\mathfrak{o}_E}) J_{\mathsf{M}}^1 n_s P(\Lambda_{\mathfrak{o}_E}) J_{\mathsf{M}}^1 = P(\Lambda_{\mathfrak{o}_E}) J_{\mathsf{M}}^1 n_w n_s P(\Lambda_{\mathfrak{o}_E}) J_{\mathsf{M}}^1.$$

Since $P(\Lambda_{\mathfrak{o}_E})$ and $n_s = n_{s_i}$, with $i \in \{2, \dots, m\}$, are both contained in $P(\Lambda_{\mathfrak{o}_E}^{\mathsf{M}})$, these normalize J_{M}^1. Hence the left-hand side is equal to

$$P(\Lambda_{\mathfrak{o}_E})J^1_{\mathsf{M}}n_w P(\Lambda_{\mathfrak{o}_E})n_s P(\Lambda_{\mathfrak{o}_E})J_{\mathsf{M}} = \bigcup_v P(\Lambda_{\mathfrak{o}_E})J^1_{\mathsf{M}}n_v P(\Lambda_{\mathfrak{o}_E})J^1_{\mathsf{M}},$$

where $v \in \boldsymbol{W}_m$, and n_v denotes its a representative in $N_{\mathfrak{b}}$. By [2, (1.6.1)] and [16, Theorem 2.3], we have

$$P(\Lambda_{\mathfrak{o}_E})J^1_{\mathsf{M}}n_w P(\Lambda_{\mathfrak{o}_E})n_s P(\Lambda_{\mathfrak{o}_E})J^1_{\mathsf{M}} \cap B^\times = P(\Lambda_{\mathfrak{o}_E})n_w P(\Lambda_{\mathfrak{o}_E})n_s P(\Lambda_{\mathfrak{o}_E}).$$

Hence it is enough to prove that $n_w P(\Lambda_{\mathfrak{o}_E})n_s \subset P(\Lambda_{\mathfrak{o}_E})n_w n_s P(\Lambda_{\mathfrak{o}_E})$.

For $\widetilde{\Phi}_{\mathfrak{b}} \supset \widetilde{\Delta}_{\mathfrak{b}}$ in Section 2, write $\alpha = \widetilde{\alpha}_i = \alpha_i \in \widetilde{\Delta}_{\mathfrak{b}}$. Then $s = s_\alpha$. Let $\widetilde{\Phi}^+_{\mathfrak{b}}$ be the subset of positive roots in $\widetilde{\Phi}_{\mathfrak{b}}$ spanned by $\widetilde{\Delta}_{\mathfrak{b}}$. We have $s(\alpha) = -\alpha$ and $s(\widetilde{\Phi}^+_{\mathfrak{b}} - \{\alpha\}) \subset \widetilde{\Phi}^+_{\mathfrak{b}}$. For $w \in \boldsymbol{W}_m$, set $\widetilde{\Gamma}_w = \{\beta \in \widetilde{\Phi}^+_{\mathfrak{b}} \mid w(\beta) < 0\}$. Then it is well known that $\ell(w)$ is equal to $^\#(\widetilde{\Gamma}_w)$, the cardinality of the set $\widetilde{\Gamma}_w$. The assumption $\ell(ws) > \ell(w)$ is equivalent to $w(\alpha) > 0$, as in the proof of Proposition 3.5. Hence $\alpha \notin \widetilde{\Gamma}_w$, but $\alpha \in \widetilde{\Gamma}_{ws}$. We have $s(\widetilde{\Gamma}_w) \subset \widetilde{\Gamma}_{ws}$. It follows that

$$\widetilde{\Gamma}_{ws} = s(\widetilde{\Gamma}_w) \amalg \{\alpha\}. \tag{4.4}$$

This implies $\ell(ws) = \ell(w) + 1$ (cf. [10, 1.5]). We have an Iwahori–Bruhat decomposition $G_E = P(\Lambda^{\mathsf{m}}_{\mathfrak{o}_E})\boldsymbol{W}P(\Lambda^{\mathsf{m}}_{\mathfrak{o}_E})$ of G_E. Denote by ℓ_{m} the length function of the affine Weyl group $\boldsymbol{W}$ (cf. [13, 5.1]). Let $\Phi_E \supset \Delta_E$ be the root system of G_E and the basis with respect to the maximal E_0-torus defined by the basis $\mathcal{V} = \{e_i \mid i \in \widetilde{I}\}$ in Section 2. We have $\boldsymbol{W}_m = \boldsymbol{W}_{\mathfrak{b}} \subset \boldsymbol{W}$. So we can define $\Gamma_w = \{a \in \Phi^+_E \mid w(a) < 0\}$ for $w \in \boldsymbol{W}_m$ like $\widetilde{\Gamma}_w$. Then the equality (4.4) also shows $^\#(\Gamma_{ws}) = {}^\#(\Gamma_w) + {}^\#(\Gamma_s)$. Hence we obtain

$$\ell_{\mathsf{m}}(ws) = \ell_{\mathsf{m}}(w) + \ell_{\mathsf{m}}(s). \tag{4.5}$$

Let $\widetilde{M}_{\mathfrak{b}} = \mathfrak{b}_0 \cap \widetilde{M}$ be as before, and set $M_{\mathfrak{b}} = \widetilde{M}_{\mathfrak{b}} \cap G$. Then $M_{\mathfrak{b}} = P(\Lambda_{\mathfrak{o}_E}) \cap M$ and $P(\Lambda_{\mathfrak{o}_E}) = M_{\mathfrak{b}}P(\Lambda^{\mathsf{m}}_{\mathfrak{o}_E})$. Since the group $\boldsymbol{W}_m = \boldsymbol{W}_{\mathfrak{b}}$ normalizes $M_{\mathfrak{b}}$. It thus follows from (4.5) that

$$\begin{aligned} n_w P(\Lambda_{\mathfrak{o}_E})n_s &= n_w M_{\mathfrak{b}} P\big(\Lambda^{\mathsf{m}}_{\mathfrak{o}_E}\big)n_s = M_{\mathfrak{b}} n_w P\big(\Lambda^{\mathsf{m}}_{\mathfrak{o}_E}\big)n_s \\ &\subset M_{\mathfrak{b}} P\big(\Lambda^{\mathsf{m}}_{\mathfrak{o}_E}\big)n_w n_s P\big(\Lambda^{\mathsf{m}}_{\mathfrak{o}_E}\big) \subset P(\Lambda_{\mathfrak{o}_E})n_w n_s P(\Lambda_{\mathfrak{o}_E}). \end{aligned}$$

The proof is completed. □

We are now ready to state the main theorem as follows:

Theorem 4.3. *Let notation and assumptions be as above. Then the Hecke algebra $\mathcal{H}(G, \lambda_P)$ is generic, that is, it has a basis T_w, for $w \in \boldsymbol{W}_m$, as a $\mathbb{C}$-vector space, that satisfy the following multiplication relations: For $w \in \boldsymbol{W}_m$ and $i \in \{0, 1, \dots, m\}$, we have $T_w * T_{s_i} = T_{ws_i}$ if $\ell(ws_i) > \ell(w)$, and*

$$T^2_{s_i} = a_i T_{s_i} + b_i 1, \quad a_i, b_i \in \mathbb{C},\ b_i \neq 0$$

with $(a_i, b_i) = (q^f_E - 1, q^f_E)$ for $i \in \{1, \dots, m-1\}$.

Proof. Let $w \in \boldsymbol{W}_m$. If $w = s_{i_1} \cdots s_{i_d}$ is a reduced expression, by Proposition 4.2, we inductively obtain $T_w = T_{s_{i_1}} * \cdots * T_{s_{i_d}}$. Hence $\mathcal{H}(G, \lambda_P)$ is an associative $\mathbb{C}$-algebra generated by T_{s_i}, for $i \in \{0, 1, \dots, m\}$. The quadratic relations have been seen above. □

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